Mathematical Modeling: Bark Beetle “X”

Introduction

Basics

- In mathematical modeling, we attempt to construct one or more mathematical or logical formulas which describe how a “real world” system behaves.
- Similar in many ways to the classroom practice of solving “story problems”, mathematical modeling involves identifying the constants and variables for the different factors and quantities in the system.
- Usually, time is a primary element of such models, since we are interested in how the system changes over time.
- Mathematical modeling almost always involves significant simplification of the real world system, since it is usually impractical (if not impossible) to articulate the full complexity of the system in mathematical formulas.

Bark Beetle Example

- Since the intent here is to illustrate the process of constructing a mathematical model, we will begin with a hypothetical beetle species (already somewhat simplified from any actual species), and simplify from there.

Basic Characteristics of Bark Beetle “X”

Life Cycle

- Developmental cycle (inside tree) lasts approximately ½ year (depending on ambient temperature), from the laying of eggs to the adult beetle emerging from tree.
- After emerging, the female searches for the next host tree.
- Beetles tend to fly in a more or less straight line (barring obstacles, or a new tree found); prevailing winds affect the distance/direction.
- If no suitable host tree is found, the beetle generally dies within a week or two; in that time, in the absence of wind, the beetle can travel as far as 1 km from its starting point.
- When suitable tree is found, female emits a pheromone which attracts males in the vicinity.
- Male/female pairs burrow under the bark to construct egg gallery.
- Each female lays 50-100 eggs, depending on the ambient temperature.

Host Trees

- Preferred trees are those which already weakened (e.g. via lightning strikes, other blights, etc.), or which have recently fallen; the resinous sap of healthy trees is a natural defense against burrowing attack. (Where a few pairs of beetles can successfully infest a weak tree, hundreds of beetles may be necessary for success against a healthy tree.)
- Trees which are already infested soon become less attractive to successive waves of beetles.
- As they burrow, the beetles spread spores of a fungus which impacts the tree’s ability to carry water and nutrients; burrowing by the larvae further damages the tree.
- As larvae burrow out of the tree, they pick up spores of the fungus, to carry to the next tree.
- After a successful infestation, the tree is virtually doomed; it will eventually die in almost all cases.

Common Defenses

Natural

- A healthy tree population keeps the bark beetle population generally stable, through the natural defense mechanisms of the trees.
- When the tree population is weakened more than normal – by fire, lightning, drought, etc. – bark beetle infestations can increase significantly.

Artificial

- Weak trees can be felled and left in shade before infestation, making them more attractive; after infestation (but before the new beetles emerge), these trees can be removed, and covered with transparent plastic, thus increasing the temperature to the point where the beetles inside are killed.
- Standing trees can also be felled after infestation, but removal is often more difficult.
- Because larger standing trees are more likely to act as obstacles (and thus slowing airborne dispersal), smaller diameter trees are usually selected for proactive felling
- Pheromone traps can be used to attract male beetles, diverting them from trees (male-female pairs are required to construct and populate egg galleries)

Assumptions and Simplifications

Time

- Assume a time-step interval of one week.

Beetle Life Cycle and Behavior

- Assume a single bark beetle species (“X”), with no predators.
- Ignore temperature impact on developmental cycle and number of eggs
- Ignore beetle gender; assume beetles which locate a suitable host tree send out pheromone attractor to all beetles in vicinity; total resulting beetles attacking a tree will be rounded down to nearest number of pairs.
- Assume each pair produces 75 eggs which eventually emerge from the tree as adults.
- Assume all beetles which have not found (or been attracted to) a new host tree within a week after emerging die.
- Assume beetles pick a random direction for flight, and maintain that direction for 0 to 500 meters (using the “Normal” probability distribution, with an average of 250 meters, and a standard deviation of 50 meters) during each time step; prevailing wind velocity is added to beetle velocity at each time step.
- Assume constant-velocity straight-line travel for beetles, without taking obstacles into account (to an extent, obstacles are implicitly accounted for by the range of possible velocities).

Trees

- Assume a single, generic type of tree.
- Assume all trees are standing – either healthy or weak – or recently felled (this will be useful when we use our model to explore effectiveness of some of the defenses mentioned above).
- Initially, all trees will be considered to be living and not infested.

Infestation

- Assume only 1 pair of beetles is required for a successful infestation of a weak tree.
- Assume 100 pairs of beetles required to overwhelm & infest a healthy tree.

- Assume that a beetle attack on a tree continues for one time step; after this, a successful infestation produces an infested tree (which will not be selected as a suitable host in the future), while an unsuccessful infestation results in the death (and lack of further beetles produced) of the infesting beetles (a tree which is the target of an unsuccessful infestation can be the target of future infestation attempts).

- Ignore carrying capacity of trees. In other words, we are not (yet) taking into account that while all of the offspring of a few pairs of beetles will have enough nutrients to emerge from a tree, the offspring of hundreds of beetles that burrow into a single tree might very well not survive to emerge.

Other Environmental Factors

- Assume constant wind direction and speed.

Other Model Factors

- Assume an area interval (i.e. resolution) of one hectare (10,000 m^2).

- Assume that all of the beetles in the area are involved (initially or following the pheromone signal) in the attacks, and that the beetles are approximately evenly distributed among the trees (whether healthy or weak).

Formulas

Adult Beetle Emergence

Here, we will construct a formula which predicts the number of adults beetles which will emerge from the tree, based on the number of eggs which were laid in the galleries, 26 weeks previously.

$$E_t = 75P_{t-26}$$

where

$E_t =$ Adults emerging at time t

$P_{t-26} =$ Pairs participating in successful infestations at time $t-26$

(For the formulas below, there is an implied time subscript, which is left out for notational simplicity.)

Travel Distance

As described in the assumptions, after emerging from the tree, adult beetles will fly in a random direction, for a distance between 0 and 500 meters (since our basic linear unit of measurement is 100 m (a hectare is 100 m X 100 m), we must adjust accordingly).

$$\mathbf{S} = \mathbf{S}_F + \mathbf{S}_W$$

$$\left|\mathbf{S}_F\right| = N(2.5,0.5)$$

where

$\mathbf{S} =$ Displacement vector

$\mathbf{S}_F =$ Displacement due to the beetle's flight, absent wind

$\mathbf{S}_W =$ Displacement due to wind

$\left|\mathbf{S}_F\right| =$ Magnitude (distance) of flight displacement vector, absent wind

$N(2.5,0.5) =$ Random value, distributed according to the Normal distribution, with a mean of 2.5 and a sta

Number of New Attacks

Here, we are assuming that if there are enough beetles in the area to attack every tree with at least one pair of beetles, then every tree will be attacked; otherwise, as many trees will be attacked as there are pairs of beetles in the area.

$$
\begin{aligned}
A &= W + H \text{ (if } W + H < \frac{B}{2}\text{)} \\
&= \left\lfloor \frac{B}{2} \right\rfloor \text{(if } W + H \geq \frac{B}{2}\text{)}
\end{aligned}
$$

$$A_H = \left\lfloor \frac{AH}{W+H} \right\rfloor$$

$$A_W = A - A_H$$

where

$A =$ Total attacks (in the given time interval)

$A_W =$ Attacks on weak trees

$A_H =$ Attacks on healthy trees

$W =$ # of weak trees in the area

$H =$ # of healthy trees in the area

$B =$ # of beetles in the area (after taking into account airborne travel)

Number of Infestations (Successful Attacks)

Finally, we construct formulas for the number of successful attacks, and the number of pairs of beetles involved in those attacks. Once we have that, we will know how many beetles will emerge, 26 weeks in the future (these will then fly to different areas, and attack more trees, and so on).

$$\overline{P} = \frac{B}{2A}$$

Then, if $\overline{P} < 1$ (i.e. fewer than one pair of beetles per tree)

$$I_H = 0$$
$$P_H = 0$$
$$I_W = \lfloor A_W \overline{P} \rfloor$$
$$P_W = I_W$$

Or, if $1 \leq \overline{P} \leq 99$ (i.e. not enough beetles for successful attacks against healthy trees), then

$$I_H = 0$$
$$P_H = 0$$
$$I_W = A_W$$
$$P_W = \lfloor A_W \overline{P} \rfloor$$

But, if $99 < \overline{P} < 100$ (i.e. some attacks against healthy trees will be successful), then

$$I_H = \lfloor A_H (\overline{P} - \lfloor \overline{P} \rfloor) \rfloor$$
$$P_H = 100 I_H$$
$$I_W = A_W$$
$$P_W = \lfloor A_W \overline{P} \rfloor$$

Finally, if $\overline{P} \geq 100$ (i.e. all attacks will be successful), then

$$I_H = A_H$$
$$P_H = \lfloor A_H \overline{P} \rfloor$$
$$I_W = A_W$$
$$P_W = \lfloor A_W \overline{P} \rfloor$$

In all cases,

$$I = I_H + I_W$$
$$P = P_H + P_W$$

Where

$B =$ Number of beetles in the area
$A =$ Number of attacks against trees (both healthy and weak)
$\overline{P} =$ Average number of beetle pairs per attack
$I_H =$ Successful infestations of healthy trees
$P_H =$ Pairs of beetles participating in successful infestations of healthy trees
$I_W =$ Successful infestations of weak trees
$P_H =$ Pairs of beetles participating in successful infestations of weak trees
$I =$ Total successful infestations
$P =$ Total pairs of beetles participating in infestations

Next Steps

Implementation

- We can implement the above model in a variety of runtime environments (e.g. Java, C/C++, StarLogo, etc.

Validation

- If we were modeling the behavior of an actual species of bark beetle, we would compare the results produced by running an implementation of the model with actual data, to assess whether our model is close enough to the real world system to be useful.

Validating With Different Initial Conditions

- While continuing to validate the model vs. the real world system, we would look at the behavior of the model under different initial conditions (e.g. distributions of healthy and weak trees, prevailing winds, initial population and placement of bark beetles). Without doing this, we would be unable to speak confidently about the model's ability to predict real-world behavior across the spectrum of likely scenarios.

Model Revision

- If the behavior of the model implementation doesn't come close enough to the real world for our purposes, we must then revisit the model, to determine what additional factors we should incorporate, what assumptions should be revised, etc. (Of course, it might also be that we have calculation or implementation errors; these should also be detected and corrected in the validation and revision process.)

Scenario Analysis

- When we are satisfied that are model approximates the behavior of the real world system – across a range of input conditions – closely enough to be useful, we can then begin to examine alternative scenarios, such as the use (and effectiveness) of tree felling strategies to combat infestation.