Fractal Illuminations

New Mexico
Supercomputing Challenge
Final Report
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Statement:

Our challenge is to further investigate the sequence of fractal images developed within the Mandelbrot set by changing the exponential power of the set to a fraction. In this problem we will specifically focus on the Mandelbrot solution of the Julia set fractals. These fractals are geometric figures that repeat themselves (are self-similar) under all levels of magnification. The particular fractal set we are working with are Julia set fractals, narrowing in on the Mandelbrot solution of that set, which is found through the formula: $(a+bi)^2 + (c)$.

Self-similar geometric figures are found everywhere in nature, from the surface of the Moon, to a stalk of broccoli. Even in tree braches we see how self-similar, fractal-like patterns are created. By applying that pattern, or set, to a tree, scientists can mathematically determine how much CO_2 is captured in that tree, and by expanding the set, the amount of carbon captured by the whole forest can be estimated.

Fractal patterns have other uses in the natural sciences, and indeed this is a rapidly growing area of study. For example, human blood vessels mimic fractal patterns, naturally taking advantage of a fractal's inherent efficiency, due to a near-infinite surface area, for transporting oxygen and nutrients to all of the body's cells. These healthy cells appear in quite different patterns, compared to the cancerous cells, in someone with cancer.

These once invisible, though pervasive, geometric figures were made visible with the help of computers. By using the processing power of computers to graphically plot a fractal set, a process that would take days manually, fractal patterns can now be examined and studied. Fractal images are being used for more than mere intellectual exercises, however. By using fractals, programmers can create more realistic computer-generated mountains by using the self-similar nature of the fractal set to create natural appearing "roughness", making the mountains look more realistic. Perhaps the most famous, or at least most widely viewed, use of fractal images to create natural-looking terrain are the three most recent *Star Wars* films. Virtually all of the sets were "virtual", in that they existed only within the computers of Industrial Light and Magic. The "lava scene", which is the climax of *Star Wars, Episode 3, The Revenge of the Sith* was really just a beautiful and exciting use of fractal mathematics (if only Lucas took as much care with the plot…).

Our problem is to change the Mandelbrot set by altering the exponent in which it is found, to a fractional power. Using the software Xaos, when we ran the Mandelbrot set with integer exponents from 1 to 6, the sequence of the sets appear to mimic cellular mitosis. This brings us to our hypothesis, that when the Mandelbrot set is run to fractional powers we predict that the resultant images will demonstrate mimicry of cellular mitosis.

Description of method:

Julia set fractals exists on the real and imaginary number plane. So complex numbers are in the form of a binomial (real + imaginary) i.e.: (3+5i). We will generate a thirty-frame per-second film that will show the morphing of the Mandelbrot set formula when moving the exponent fractionally from one to four. We will color the exterior tested locations based on the speed the tested location moves away from the set. The problem is you can't raise a binomial (a+bi) to a fractional power with normal high school algebra so we use DeMoivre's theorem.

DeMoivre's Theorem, named after Abraham de Moivre, states that for any complex or real number, x and any integer: (cosx + *i* sinx)^n = cos (nx) + *i* sin (nx). This is important because it connects complex numbers with trigonometric identities (θ,d). This allows us to take a complex number and raise it by the ¾ power or any real number.

DeMoivre was born in 1667 in France. His family then moved to England when he was a little boy. When reading Isaac Newton's <u>Principia Mathematica</u> as an adult, it was this work that had inspired him to work in the fields of mathematical research. He realized that Isaac Newton's work was a breakthrough in the fields of mathematics and science.

By using De Moivre's Theorem, this lets us raise the complex numbers by converting the location to be tested to circular coordinates i.e.: (3+5i) becomes angle Θ = $\tan^{-1}$ because tangent of an angle is opposite over adjacent. So for Θ = tan-x/y, and r (the distance from [Ø, Ø]) is found by using the distance formula (Pythagorean Theorem) i.e.: r = sqrt $(3)^2 + (5)^2$. Now Θ, r, is angle, distance. De Moivre's theorem says for complex numbers in the form of (x +yi) = z, r = sqrt (x^2+y^2),

$\Theta = \tan^{-1}(x/y)$, $z^n = ((r^n)(\cos(n\Theta)) + (r^n)(\sin(n\Theta))i)$

Real part Imaginary part

If z is the complex number Mandelbrot set is found by $z^2 + c$, where c is the original position.

The equation (cosx + *i* sinx)^n = cos (nx) + *i* sin (nx) will work for the unit circle. If we wanted to use a point like (2,4i) out of the unit circle, and raise it to the n^{th} power, then you need to find the angle (θ) and the distance (d) so if:

$(2,4i)^2 = (2 + 4i)(2 + 4i) =$
$4 + 16i + 16i^2 = 4 – 16 + 16i =$
$(-12 + 16i)$.
Then,
$r = \sqrt{a^2 + b^2}$, $r = \sqrt{2^2 + 4^2}$, $r = \sqrt{4 + 16}$, $r = \sqrt{20}$,
Real: $(\sqrt{20})^2 * (\cos(\tan^{-1}(2))(2)) = -12$,
Imaginary: $(\sqrt{20})^2 * (\sin(\tan^{-1}(2))(2)) = 16$

Results of study:

During our study we have learned that our prediction was incorrect. We used DeMoivre’s theorem to expand our Mandelbrot set to the powers of 1.01 to 4 with .01 increments. This is important because as you raise the Mandelbrot from 1.01 to 4, you see that the Mandelbrot set starts from a small speck, expanding form the right clockwise, and has a “shelf” that is a product of the exponent less than 2. At the power of 2, it becomes the regular Mandelbrot set. At the power of 3, there are 2 lobes that make up the set and at 4, there are 3 lobes. In between the powers of 3 and 4 the “shelf” appears and spins clockwise to complete the next powers lobe.The short video of our output can be found at: http://chileped.home.comcast.net/~chileped/Fractal_Video.mov The video shows the change in the set from 1.01 to 4.00 in hundredths of a power.

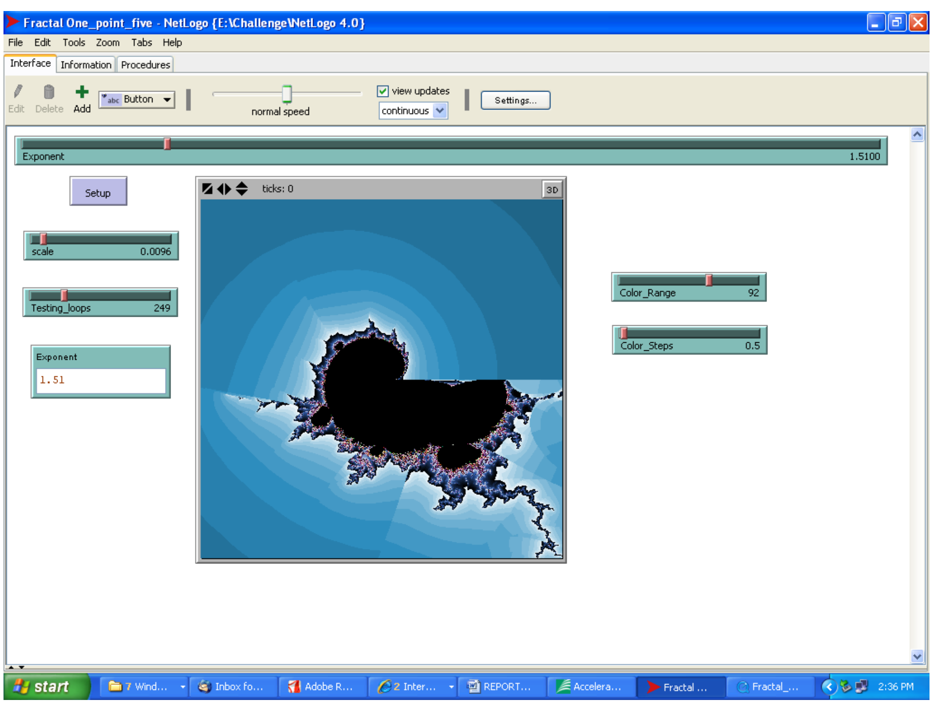
This thought that a cellular mitosis model would be created did not occur.

Conclusion:

Though our prediction was erroneous, we did however find out that fractals are mathematical models of structures found in nature all around us. Even before Benoît Mandelbrots research, many people had noticed these patterns – the organized roughness that occurs in nature. This means that, for example, genetic coding of say, human blood vessels are designed to efficiently allow blood to all of the body with room to spare. Our most significant achievement was learning how to make the program and understanding the entirety of the math that went with it. Another great discovery was taking out Mandelbrot set model to the power of -2, which yielded very interesting results.

Supporting information / Model Code:

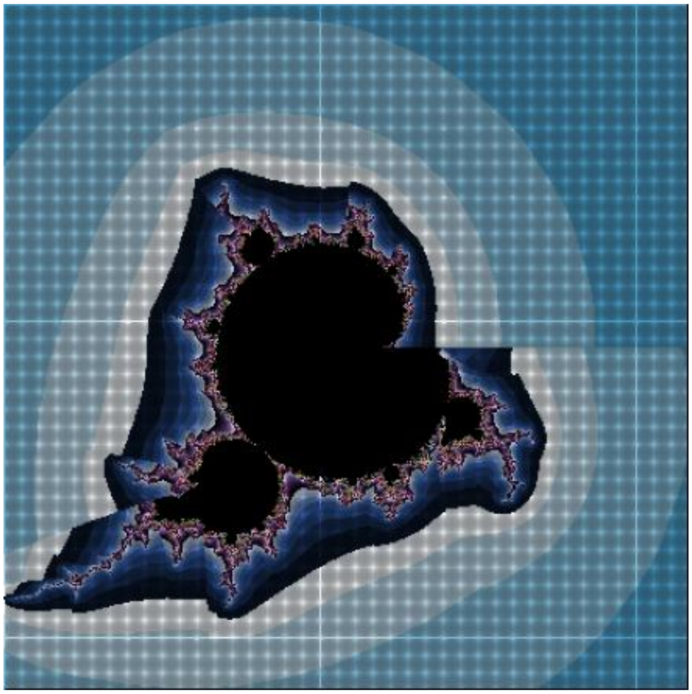
For this project we used the program Netlogo. Net logo is a simple educational tool that allows with simple syntax to make a model of a physical situation or geometric series. Below is a screen shot of the Interface page.



The code that we used to make all the different Mandelbrot pictures was:

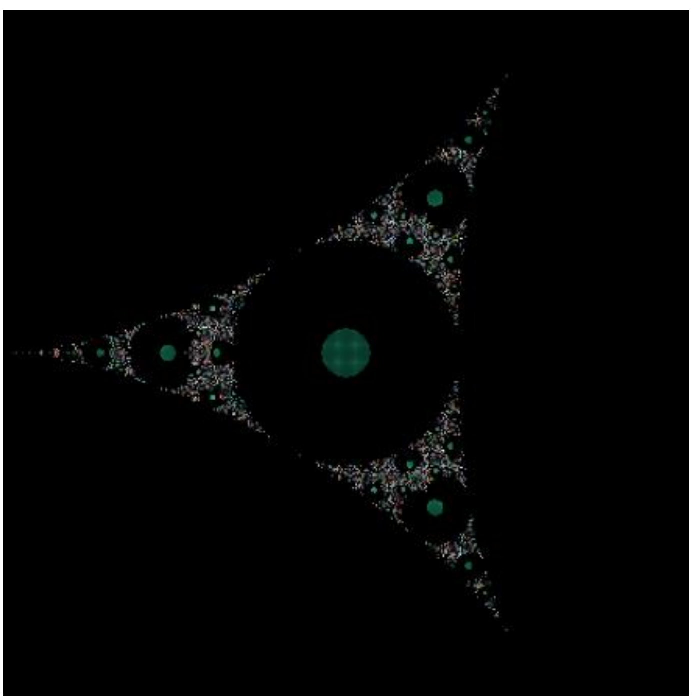
```
turtles-own
[  z-real  ;real coordinate.
   z-imag  ;imaginary coordinate.
   c-real  ;holds real and imag position.
   c-imag
   ar  ;in polar this is radius from (0,0).
   tda  ;in polar this is theta.
   counter  ;counts number of times before reaching the test limit for "not in set".
   ]
to setup
  clear-all  ;clears graphic screen, prepares program
 crt 1 [setxy -200 200 set color 1]  ;sets turtle one at (-200,200) on a 400 by 400 matrix
with a color of 1
 ask turtles [set heading 90 repeat 400 [ repeat 400 [testing fd 1] set xcor (xcor - 400) set
ycor (ycor - 1) ]]  ;tells turtle to test one point and move over 1 (90 degrees) 400 times,
then goes back <-400,-1> and repeats./ checks every point on the screen.
end

to testing
  set z-real (xcor * scale)  ;shrinks scale and holds the tested place
  set c-real z-real  ;sets original position holder
  set z-imag (ycor * scale)  ;shrinks scale and holds the tested place
```

```
  set c-imag z-imag  ;sets original position holder
  set counter Color_ Range  ;sets counter at current color choice on slider
  repeat Testing_loops [ set counter (counter + Color_Steps ) DeMoivre calc if distancexy
( z-real / scale ) ( z-imag / scale ) > ( 2 / scale ) [ set color counter stamp stop ] ]
  set color black stamp  ;testing loop
end
to DeMoivre
  if z-real = 0 [set z-real z-real + .000000000001 ]  ;avoids dividing by 0
  set ar ( sqrt((z-real) ^ 2 + (z-imag) ^ 2 ))  ;finds distance from origin in polar coordinates
  set tda ( atan z-imag z-real )  ;finds theta from coordinates
end
to calc
  set z-wreal (((ar) ^ Exponent ) * (cos( Exponent * tda )) + c-real)  ;calculation of new
real variable position
  set z-wimag (((ar) ^ Exponent ) * (sin( Exponent * tda )) + c-imag)  ;calculation of
imaginary variable position
end
```



This is the Mandelbrot raised to the 1.78^{th} power.

We also found out that when you raise the set to a negative such as -2, the set comes out like this:



Dedications:

We owe a very special thanks to Joe Vertrees, our math teacher and mentor throughout this project. Richard Foust, also one of our teachers at Freedom High, has helped us as well. Abraham de Moivre, deserves our thanks for his theorem which to a great extent, made our project possible. Benoît B. Mandelbrot, credited for the discovery of the Mandelbrot set, is the reason for why we did this project on fractals. His fascinating, modern discovery is what led us to continue on his fascinating research.

References:

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